Partial amplitude death in coupled chaotic oscillators

Weiqing Liu, Jinghua Xiao, and Junzhong Yang*
School of Science, Beijing University of Posts and Telecommunications, Beijing, 100876, People's Republic of China
(Received 26 January 2005; revised manuscript received 11 April 2005; published 8 November 2005)

We have investigated the dynamics of the coupled Lorenz oscillators numerically and theoretically. We find the partial amplitude death when the interaction is strong enough. The linear stability analysis of the partial amplitude death is proposed.

DOI: 10.1103/PhysRevE.72.057201 PACS number(s): 05.45.Xt, 07.50.Ek, 84.30.Bv

I. INTRODUCTION

Coupled oscillators are frequently encountered in electrical engineering, computational biology, and physics [1]. The applications are found in coupled laser systems, Josephson junctions arrays, electrical circuits, etc. [2–4]. Since the Pecora's work in 1990 [5], the coupled chaotic oscillators have been a hot spot. Lots of phenomena related to the synchronous chaos have been found in coupled chaotic systems, for example phase synchronization, partial synchronization, riddled basin, kinds of bifurcations of the synchronous chaos, and so on [6–10]. However there are some coupled systems which cannot realize chaos synchronization even when the interaction among the elements are strong enough [9]. It is interesting to see whether such systems may still display rich dynamic behaviors. In this paper, we will investigate such a coupled system where the chaos synchronization is impossible.

The model used in this paper is a coupled Lorenz oscillators. The isolated system in dimensionless form is described as

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - 10xz, \quad \dot{z} = 2.5xy - bz, \tag{1}$$

which is used to describe an analog circuit reported in Ref. [11] where dimensionless parameters $\sigma=10.19$, $b=2.664$, and $r=28.17$. When variable y is coupled to the equation of z, the chaos synchronization is impossible no matter how strong the interaction is [12]. We first explore the dynamics of the coupled system with two identical Lorenz oscillators numerically. We find that the translational symmetry in the system is broken if the interaction becomes strong enough where one element oscillates with large amplitude and the other with small amplitude. Further increasing the strength of the interaction, we find that the small amplitude oscillation collapses to a partial amplitude death state where part of the variables of one oscillator stays at rest. Then we propose a linear stability analysis for the partial amplitude death state.

The paper is organized as following. In Sec. II, we present the numerical results. In Sec. III, we give a linear stability theory for the partial amplitude death. Finally, a brief summary is made in Sec. IV.

*Author to whom correspondence should be addressed. Electronic address: jzyang@bupt.edu.cn

II. NUMERICAL RESULTS

The coupled system can be described by the following equation:

$$\begin{aligned} &\dot{x}_1 = \sigma(y_1 - x_1), \quad \dot{y}_1 = rx_1 - y_1 - 10x_1z_1, \\ &\dot{z}_1 = 2.5x_1y_1 - bz_1 + \epsilon(y_2 - y_1), \quad \dot{x}_2 = \sigma(y_2 - x_2), \\ &\dot{y}_2 = rx_2 - y_2 - 10x_2z_2, \quad \dot{z}_2 = 2.5x_2y_2 - bz_2 + \epsilon(y_1 - y_2). \end{aligned} \tag{2}$$

The coupled system has translational symmetry between the oscillators 1 and 2. However the reflection symmetry under the transformation $x \to -x$, $y \to -y$, and $z \to z$ in a single Lorenz system is broken. The fourth-order Runge-Kutta algorithm is used to integrate Eq. (2) with time step of 0.01.

The dynamics of the coupled oscillators is controlled by the coupling constant ϵ. The phase portraits projected in the x-z plane for different ϵ are shown in Fig. 1. Once the coupling between the oscillators is switched on, the reflection symmetry in any single Lorenz oscillator is broken. With the increase of ϵ, one wing of the attractor expands [for example, Figs. 1(a) and 1(b)]. However the translational symmetry between the two oscillators are still kept. Further increasing the coupling constant beyond a critical coupling constant, the two wings structure of the attractor disappears: only the expanding one survives. The translational symmetry is broken either and two oscillators stay at different attractors. The process of the transition can be found in Figs. 1(c) and 1(d). When the coupling constant is strong enough, an interesting state is found where one of the attractors shrinks to a line parallel to the z axis in Fig. 1(e). It means that x_2 becomes independent of time while z_2 not.

To gain more knowledge about the behaviors described in Fig. 1, we record the time sequence of x. Before the translational symmetry is broken, the two oscillators jump between two wings of the attractor while not in synchronization [Figs. 2(a) and 2(b)]. A finding not reflected in Fig. 1 is that the two oscillators with the same attractor behave quite differently for certain range of ϵ where one will oscillate with large amplitude if the other oscillates with small amplitude, which is shown in Fig. 2(c). Each oscillator jumps intermittently between the oscillations with the small and large amplitude. With the increase of the coupling constant, the jump between the two kinds of oscillations becomes less frequent and eventually one oscillator stays on the large amplitude oscillation while the other on the small one [for example in Fig. 2(d)].

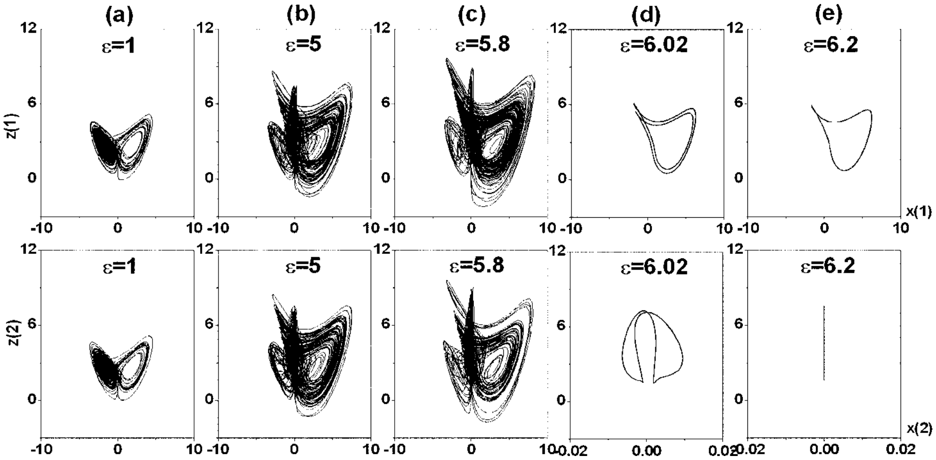

FIG. 1. The phase trajectories in the x-z plane are recorded for different ϵ from (a) to (e). Top panels are for the oscillator 1 and the bottom for the oscillator 2.

The disappearance of the intermittency signals the broken translational symmetry in the solutions of Eq. (2). The temporal behavior of the trajectory in Fig. 1(e) is shown in Fig. 2(e) where the small oscillation of variable x_2 dies off while x_1 continues the oscillation with large amplitude. It is worth noting that the oscillator 2 only stay at rest for the variables x_2 and y_2 while the variable z_2 still oscillates with the amplitude comparable to the oscillator 1 and the value of z_2 increases with ϵ (not shown here). In contrast to the amplitude death found in coupled limit cycles [13,14] where the amplitude death indicates that all elements stay at quenched state, we term *the phenomenon of the amplitude death in some dynamical variables as a new type of partial amplitude death*. A different partial amplitude death has been mentioned in Ref. [14], which refers to a phenomenon where some of the elements die off while the rest keeps oscillating. Especially, the requirement of the nonidentical elements or the delayed interaction among elements for the amplitude death (or the partial amplitude death) in Refs. [13,14] is not required in this paper. One thing to be noted is that the partial amplitude death with $x_2=y_2=0$ has a partner with $x_1=y_1=0$ according to the translational symmetry of Eq. (2). Depending on the initial condition, the system will evolve into one of them.

Moreover, the bifurcation diagram versus coupling constant ϵ for each oscillator is presented in Fig. 3 where the partial amplitude death in the variable x_2 when $\epsilon > 6.04$ is clear. The inset in Fig. 3(b) shows the amplitude of the small oscillation grows gradually after the partial amplitude death becomes unstable. The spectrum of the Lyapunov exponents shown in Fig. 3(c) confirms the transitions found in Figs. 3(a) and 3(b). The blowup of the spectrum in the range of $\epsilon \in (6, 6.05)$ shows that the second largest Lyapunov exponent collides with zero at $\epsilon=6.04$ and keeps negative on both sides of the transition, which indicates that the instability of the partial amplitude death is related to the period-doubling bifurcation.

The discontinuity in the size of the attractor versus ϵ in Fig. 3 at $\epsilon \approx 6.01$ indicates a crisis-induced-transition [15] which is responsible for the change of the translational sym-

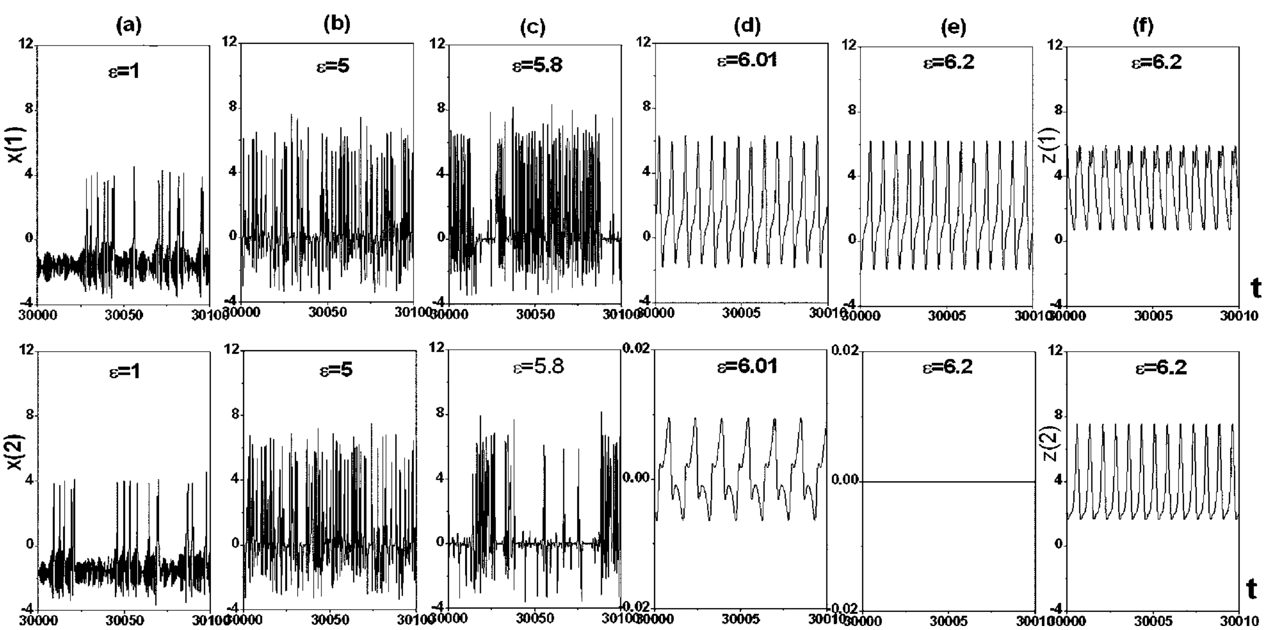

FIG. 2. The time evolutions of the variable x are recorded for different ϵ from (a) to (e). (f) The time evolution of the variable z with the same ϵ as (e). Top panels are for the oscillator 1 and the bottom for the oscillator 2.

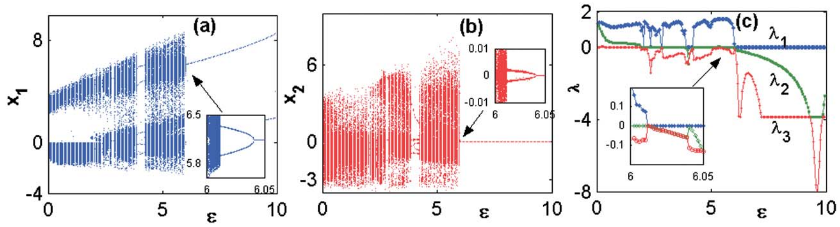

FIG. 3. (Color online) (a) and (b) The bifurcation diagrams versus ϵ for x_1 and x_2, respectively. The data is obtained when x_1 reaches its maximum. (c) The first three largest Lyapunov exponents, Λ, are plotted versus ϵ. The insets in (a), (b), and (c) enlarge the narrow parameter region for $\epsilon \in (6, 6.05)$ where the partial amplitude death loses its stability.

metry of solutions in the studied system. The crisis here is a result of the collision between the orbit of the large amplitude oscillation and the stable manifold of the origin ($x_i = y_i = z_i = 0$, $i = 1,2$) which prevents the movement of the oscillator from one wing to the other of the Lorenz attractor for $\epsilon > 6.01$. Except for the abrupt change in the size of the attractor, the crisis induces intermittency also which can be hinted in Figs. 4(a) and 4(b). Strictly, the demonstration of the crisis requires the location of the stable manifold of the origin, however it is difficult to do in a high dimensional phase space. Nevertheless Figs. 4(c) and 4(d) can give us an indirect proof on such collision. The data in Figs. 4(c) and 4(d) is obtained from Figs. 4(a) and 4(b) where one exchange between the states of two oscillators occurs. Figure 4(d) shows that the confinement to one wing by the stable manifold of the origin has been broken and the oscillator with large amplitude jumps to another wing and stay for a while before exiting to the small amplitude oscillation.

III. THEORETICAL ANALYSIS

At the first glance, the partial amplitude death seems ambiguous since x_2 and y_2 do not constitute a closed system while the variable z_2 is time-dependent. However in the sub-

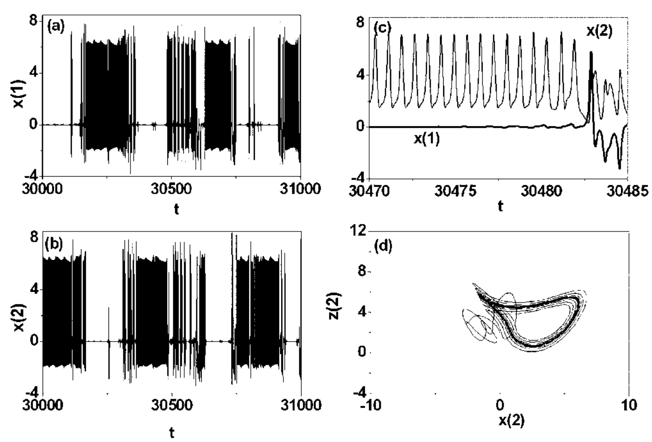

FIG. 4. $\epsilon = 5.95$ is close to the crisis-induced transition. (a) and (b) The evolutions of $x(1)$ and $x(2)$ where the intermittency is clear. (c) A time period of the evolution of $x(1)$ and x_2 is shown where one exchange between the states of the oscillators occurs. (d) The trajectory of x_2 in x-z plane during the time period in (c) where we can find that the large amplitude oscillation jumps to another wing of the Lorenz attractor.

system of x_2 and y_2, the variable z_2 appeared in the y_2 equation is multiplied by x_2. Therefore, if x_2 goes to zero, z_2 may not have any effects on the behavior of the subsystem. To explain the partial amplitude death, it is necessary to explore its stability. Let us consider the equations of x_2 and y_2 and treat them as a subsystem driven by the signal z_2,

$$\dot{x}_2 = \sigma(y_2 - x_2), \quad \dot{y}_2 = rx_2 - y_2 - 10x_2z_2. \tag{3}$$

The subsystem has unique equilibrium (0,0) which represents the partial amplitude death state. The Jacobian matrix at the equilibrium is $A = \begin{pmatrix} -\sigma & \sigma \\ r-10z_2 & -1 \end{pmatrix}$ and has two eigenvalues $\lambda_{1,2} = [\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\,\mathrm{Det}(A)}]/2$ where $\mathrm{Tr}(A) = -(\sigma+1)$, $\mathrm{Det}(A) = \sigma(1 - r + 10z_2)$. With $\sigma = 10.19$, $b = 2.664$, and $r = 28.17$, we have $\mathrm{Tr}(A) = -11.19 < 0$ and $\mathrm{Det}(A) = \sigma(10z_2 - 27.17)$. The equilibrium is stable only if $z_2 > (1-r)/10$. Since z_2 is dependent of time, the instantaneous stability of the equilibrium changes with time. We plot the instantaneous maximum growth rate, $\Lambda = \max\{\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)\}$ [$\min\{\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)\}$ is always negative], in Fig. 5(a). Three types of the equilibria can be found: the saddle with positive maximum growth rate, the stable node with negative maximum growth rate, and the stable focus with constant growth rate. The virtual stability of the equilibrium (0,0) is determined by the accumulated growth rate during the evolution. In other words, the equilibrium is stable if and only if the area enclosed by the time axis [the dashed line in Fig. 5(a)] and the curve of the maximum growth rate is negative. Such a accumulated growth rate can be quantified by the Lyapunov exponents of the subsystem Eq. (3). The driving signal z_2 in Eq. (3) is obtained by simulating numerically the equations

$$\dot{x}_1 = \sigma(y_1 - x_1), \quad \dot{y}_1 = rx_1 - y_1 - 10x_1y_1,$$

$$\dot{z}_1 = 2.5x_1y_1 - bz_1 - \epsilon y_1, \quad \dot{z}_2 = -bz_2 + \epsilon y_1. \tag{4}$$

The Lyapunov exponents of the subsystem (x_2, y_2) are shown in Fig. 5(b). The negative maximum Lyapunov exponent indicates that the partial amplitude death is stable. The onset of the partial amplitude death is in agreement with the direct numerical simulation of the original system.

Since the instability of the partial amplitude death roots at the appearance of the instantaneous saddle, the small amplitude oscillation after the instability is not caused by Hopf bifurcation but a consequence of period-doubling bifurcation as mentioned in the last section. Immediately after the instability of the partial amplitude death, the continuous growing

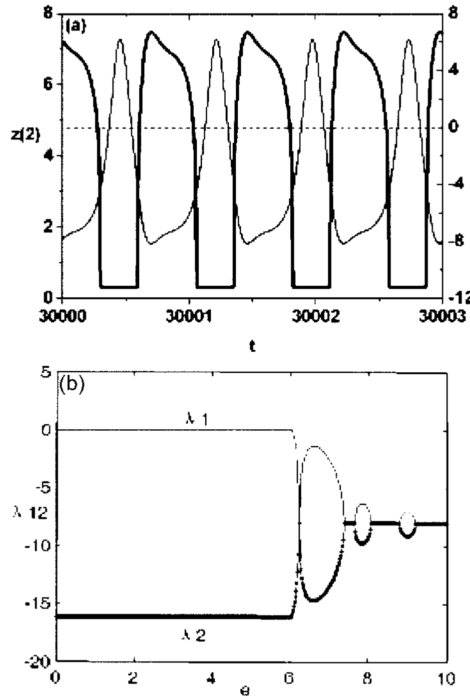

FIG. 5. (a) The time series of the instantaneous maximum growth rate Λ (bold curve) and z_2. $\epsilon=6.05$. (b) The Lyapunov exponents versus ϵ are plotted for the subsystem of x_2 and y_2. The onset of the instability of the partial amplitude death is a little larger than $\epsilon=6$.

of the amplitude from zero with the decrease of the coupling constant keeps the translational symmetry of the system broken. The translational symmetry is restored by the crisis which leads to the jump of the oscillators between the oscillations with large and small amplitude as mentioned in the last section.

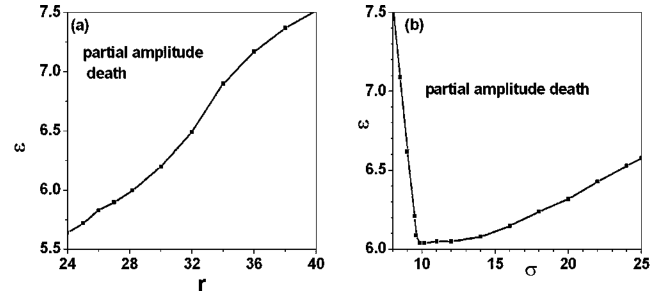

FIG. 6. The phase diagram for the partial amplitude death. (a) $\sigma=10.19$ and $b=2.664$. (b) $r=28.17$ and $b=2.664$.

The partial amplitude death is robust with the change of the system's parameters. In Fig. 6, we show the phase diagram where the partial amplitude death is stable above the curve. The reason that the partial amplitude death tends to be stable for large ϵ is that strong coupling constant usually increases the value of z_2 while we know that large z_2 favors the partial amplitude death based on the analysis above.

IV. CONCLUSION

In summary, we have investigated the dynamics of the coupled Lorenz oscillators numerically and theoretically. We find that the translational symmetry is broken when the interaction between the oscillators becomes strong. We also find the partial amplitude death when the interaction is strong enough. The linear stability analysis is presented to explain the existence of the partial amplitude death.

ACKNOWLEDGMENTS

This work was supported by Grant No. 10405004 and 10172020 from Chinese Natural Science Foundation.

[1] A. T. Winfree, *The Geometry of Biological Time* (Springer, Berlin, 1980); J. D. Murry, *Mathematical Biology* (Springer, New York, Berlin, Heidelberg, 1993); D. F. Hoyt and R. C. Taylor, Nature (London) **292**, 239 (1981); J. J. Collins and I. Stewart, Biol. Cybern. **68**, 287 (1993); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
[2] P. Hadley and M. R. Beasley, Appl. Phys. Lett. **50**, 621 (1987); S. Watanabe and S. H. Strogatz, Physica D **74**, 197 (1994).
[3] J. Guemez and M. A. Matias, Phys. Rev. E **52**, R2145 (1995).
[4] K. Wiesenfeld, C. Bracikowski, G. James, and R. Roy, Phys. Rev. Lett. **65**, 1749 (1990); C. Diorio and R. P. N. Rao, Nature (London) **405**, 891 (2000).
[5] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[6] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[7] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997); Y. Zhang, G. Hu, and H. A. Cerdeira, Chaos **13**, 1 (2003).
[8] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).
[9] J. F. Heagy, L. M. Pecora, and T. L. Carroll, Phys. Rev. Lett. **74**, 4185 (1995).
[10] G. Hu, J. Z. Yang, W. Q. Ma, and J. H. Xiao, Phys. Rev. Lett. **81**, 5314 (1998).
[11] E. Sanchez and M. A. Matias, Phys. Rev. E **57**, 6184 (1998); A. Pujol-Pere, O. Calvo, M. A. Matias, and J. Kurths, Chaos **13**, 319 (2003).
[12] J. Yang (unpublished).
[13] K. Bar-Eli, Physica D **14**, 242 (1985); D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998); R. Herrero, M. Figueras, J. Rius, F. Pi, and G. Orriols, Phys. Rev. Lett. **84**, 5312 (2000).
[14] F. M. Atay, Physica D **183**, 1 (2003); K. Konishi, Phys. Rev. E **70**, 066201 (2004); E. I. Volkov and D. V. Volkov, Phys. Rev. E **65**, 046232 (2002); F. M. Atay, in Proceedings of the 10th Workshop on Nonlinear Dynamics of Electronic Systems (Izmir, Turkey, 2002), p. 5.
[15] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, UK, 1993).